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Modelling of Rigid Rocking of Structures during Earthquake Using Linear Functions

P. Yu, S. Gutschmidt & G.A. MacRae

University of Canterbury, Christchurch, New Zealand.

ABSTRACT: New generation seismic structures are likely to suffer no damage in a major earthquake. One of the means of protecting a structure from damage is to allow it to rock. This paper describes the rocking behaviour of a simple cantilever column on various foundation types. It is shown that what is often referred to as "rocking" is in fact a combination of two modes; (i) vibration and (ii) rocking. In vibration, the supports of the structure move vertically depending on the stiffness of the supporting material and its own stiffness, but there is no lift-off. In rocking, there is no (or much less) deformation considered in the foundation material or structure, and all lateral deformation is due to lift-off. Equations are developed to describe both of these motions and while the angular displacement-time response for both modes can have some similarities, there are also major differences. The displacement-time response over one cycle looks similar in both modes; however the velocities and forces are significantly different. Also, if there is energy dissipation, the amplitudes of both modes decrease, but for rocking structures, the "period" of "vibration" also changes. An experimental study is also performed on different foundation types. It is shown that during free vibration from large displacements, rocking almost totally dominates, followed by a relatively sudden change to the vibration mode.

1 INTRODUCTION

Rocking, as a mechanism for resisting earthquake shaking, has been studied by a number of researchers (Housner 1963), (Ma 2010). A comprehensive summary of the relevant literature is given by (Ma 2010). The formulations of rocking often include the effects of soil flexibility and design procedures have been developed for these. These procedures are generally developed from concepts derived from the steady state displacement response of rocking structures. The response is then related to an equivalent vibrating SDOF structure for design.

This paper considers what is commonly termed "rocking" as two parts. The first is "true rocking" which involves lift-off of the structure from the ground with no deformation of the foundation material or structure, and "vibration" which involves no lift-off of the structure itself, but deformation of the foundation material and structure. Real structures undergo a combination of rocking and vibrating modes. The extent of these depends on the properties of the foundation material and structure itself. It can be assumed that rocking and vibrating modes occur distinctively separated from one another, and any deformed shape can be represented as a superposition of the vibration and rocking components.

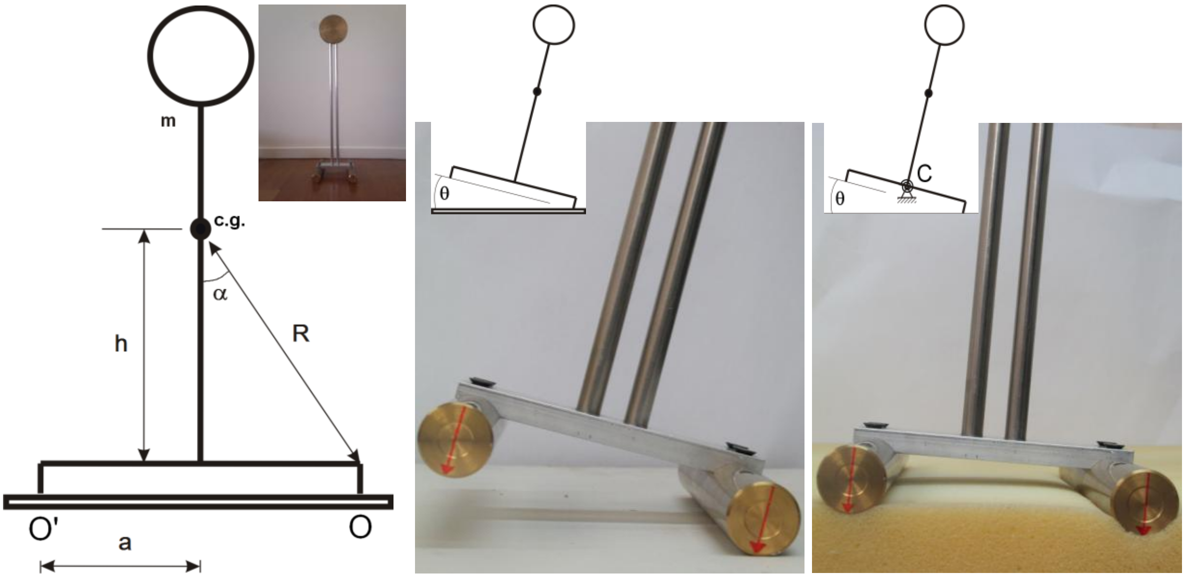
The existence of these two modes of deflections leads to the following questions which this paper attempts to answer:

1) How does the "pure rocking" response differ from the "vibration" response?
2) Is a physical model response controlled by "pure rocking", "vibration" or both?
3) What effect does the foundation flexibility have on a physical model?
4) Can the behaviour of a physical model (undergoing a combination of rocking and vibration) be presented by simple, even linear, analytical models?

The emphasis in this work is laid on looking at typically used terminologies and explaining and clarifying their meanings. Simulations are compared to experimental results and special phenomena are discussed in detail.

2 MODELS

The physical model used in this study is that given in Fig. 1a with its properties displayed in Tab. 1.



(a) Sketch and Photo (b) Rocking Model (c) Vibration Model

Figure 1: Photograph and sketch of the physical rocking structure

Table 1: Geometric and material parameters of the physical rocking structure

horizontal distance of centre of gravity to pivot O under rocking	a	0.054 m
vertical distance of centre of gravity to base of structure	h	0.205 m
shortest distance of centre of gravity to pivot O under rocking	$R=\sqrt{a^2+h^2}$	0.212 m
critical overturning angle	$\alpha = \arctan(a/h)$	14.8°
total mass of the model	m	3.40 kg
moment of inertia about pivot O	I_0	0.3268 kgm^2

Two additional analytical models are introduced, one capturing the pure rocking dynamics (Fig. 1b) and the other describing the vibrating mode of the system (Fig. 1c). The analytical model for codifying experimental rocking behaviour is the well-known Housner model, based on rigid body dynamics, (Housner 1963).

$$I_o\ddot{\theta} = -mgR\sin(\alpha-\theta) \qquad \text{for } \theta > 0$$
$$I_{o'}\ddot{\theta} = mgR\sin(\alpha+\theta) \qquad \text{for } \theta < 0 \qquad (2\text{-}1)$$

Here, θ is the angle of rocking described in Fig. 1. The pure oscillating mode of the structure is similarly modelled by a simple, linear, SDOF model with the boundary condition as shown in Fig. 1c.

$$I_C\ddot{\theta} + k_t\theta = 0 \qquad (2\text{-}2)$$

Note, that $I_o, I_{o'}, I_C$ are the moments of inertia with respect to the instantaneous centre of rotation.

3 ROCKING VERSUS VIBRATING

During rocking (Fig. 1b, Eq. (2-1)), the model will rotate about pivots O and O' in turn. In this preliminary study it is assumed that the structure is infinitely stiff, and phenomena such as sliding or bouncing are ignored for simplicity. This assumption has been justified by experimental investigations with a high-speed camera (Casio EX-F1, 300fps). Figure 2 shows one cycle of a rocking motion found from the equations of motion, neither considering energy dissipation nor possible impact mechanisms between surface and structure.

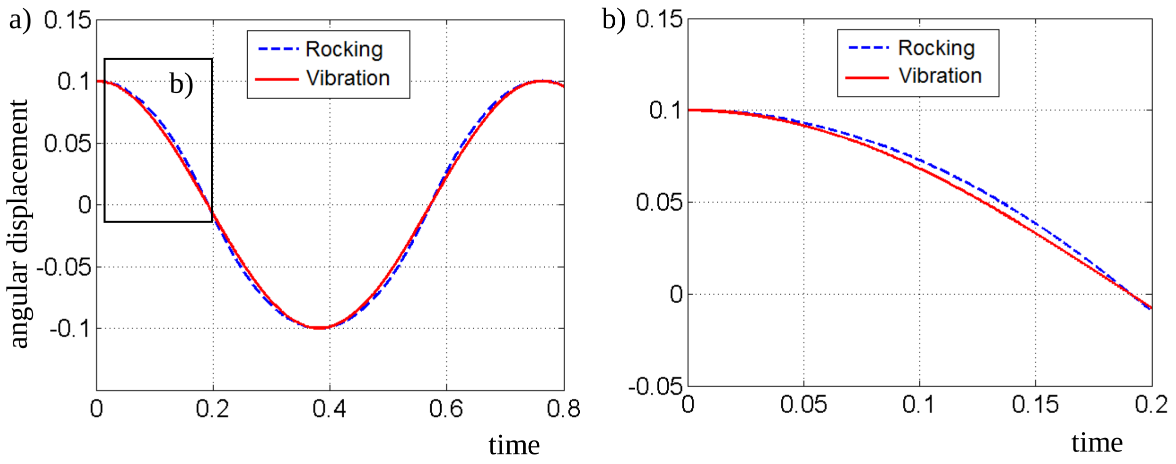


Figure 2: Comparison between a harmonic signal and the angle characteristic of a rocking behaviour a) over one cycle, b) over one quarter of a cycle

In the same Fig. 2 the rocking curve is overlaid by a harmonic (vibration) function with the same "amplitude" and "period". Although the differences in the angular displacement curves appear minor, the characteristics in velocity and acceleration look significantly different, as shown in Fig. 3.

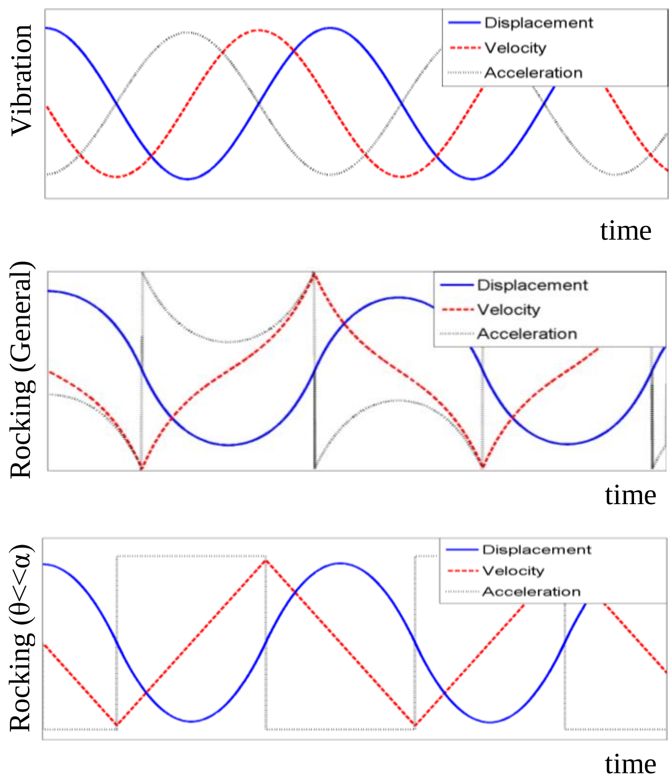


Figure 3: The angle, velocity and acceleration characteristics of a) vibration behaviour, b) rocking behaviour, large θ, c) rocking behaviour, small θ.

Rocking of a rigid structure is generated by perturbing the structure from its stable equilibrium position i.e. introducing an initial angle greater or smaller than zero. The equations of motion of the rocking structure are derived using the principle of angular momentum (Newton's second law for rotation, Eq. (2-1)).

Newton's second law is strictly applied for systems in an absolute reference frame. Rotations need to be formulated with respect to a fixed point or the centre of gravity of a system (Meriam & Kraige 2008). Under the condition that the structure is always in contact with at least one of the pivot points and furthermore that the system does not slide during the rocking process, the two pivot points O and O' can be taken as momentary fixed references for which the principle can be applied. Thus, the rocking behaviour is described by the two governing equations Eq. (2-1), depending on the direction of the angle. This immediately marks one of the differences between *rocking* – which is described by at least two differential equations and an *oscillation* – which is described by one governing equation. Furthermore, if (2-1) could be seen as equations that describe an oscillation, the linear term in theta (after linearization) would denote the "natural frequency" or the "eigenvalue" of the system, which in both equations is negative. Note, that the system in (2-1) has a positive "eigenvalue" and a complex eigenfrequency with a purely imaginary part. This implies no restoring term, and thus determines the system to be unstable, which obviously contradict the overall system's real behaviour. The explanation lies in the different natures of rocking and oscillation. The rocking process is a rotating motion that abruptly changes directions of angles. The abrupt change in direction of the angle is enforced by an impact with another structure (here the soil/ground), which abruptly changes the description of motion and boundary condition, and thus generates the alternation. The alternating behaviour of an oscillating system, in contrary, is generated by the restoring term (potential of the same structure) and not by a collision (abrupt change of set of equations) with another object. Furthermore, note that for θ being much smaller than the aspect ratio α of the structure, the right hand side of (2-1) becomes a constant and thus the overall behaviour of the structure follows an alternating free motion. The characteristic of the angle thus follows a quadratic function rather than a harmonic signal (c.f. Figs. 2 and 3c) and the acceleration signal of a rocking structure under uplift is a constant (c.f. Figs 3c and 6). This is strengthened and supported by experimental investigations (Section 4).

Thus, one has to be careful in using terminologies such as *natural frequencies/periods, restoring forces/moments, amplitudes* and *vibrations* when the rocking behaviour is to be described.

The change in "period" of a rocking and vibrating structure is investigated in what follows. Free rocking as well as vibrating behaviours are characterized by decreasing "amplitudes" over time. Furthermore one often observes an increase in frequency in those structures. In order to analyse unknown objects that can undergo either rocking or vibration (or both), systematic separate studies for rocking and vibration are performed based on fundamental knowledge of the two distinct dynamics. Maximum angular displacements, velocities, and/or accelerations as well as information on changes over time can be retrieved from experimental investigations (see Section 4). For a pure rocking behaviour, the time for a full rocking cycle TC, is often referred to as period as in (Housner 1963)

$$TC = \frac{4}{\omega_r}\cosh^{-1}\left(\frac{1}{1-\theta_0/\alpha}\right) \tag{2-3}$$

where $\omega_r, \theta_0, \alpha$ are referred to as an angular "natural frequency", initial angle at the beginning of each ¼ cycle, and the geometric aspect ratio of the structure. It is immediately obvious that the time for a rocking cycle depends on the initial condition and thus, changes over time unlike a linear oscillator where the period remains constant over time according to the relation below, where ω is the circular frequency of vibration and T_L is the period of vibration.

$$T_L = \frac{2\pi}{\omega} \tag{2-4}$$

However, for an oscillator with large amplitudes (which is a nonlinear system) the period is changing over time as well, even if damping is not considered. For the simple mathematical pendulum e.g. the closed form expression for the period is (Hagedorn 1988)

$$T_{NL} = \frac{4}{\omega}\int_0^{\pi/2} \frac{d\tau}{\sqrt{1-k^2\sin^2\tau}} \qquad \text{where } k=\sin(\theta_0/2) \tag{2-5}$$

Or for small angles ($\sin(\theta_0/2)=\theta_0/2$) the approximation

$$T_{NL} \approx \frac{2\pi}{\omega}\left(1+\frac{\theta_0^{\,2}}{16}\right) \tag{2-6}$$

By comparing Eqs. (2-3) to (2-6), as depicted in Fig. 4, it becomes immediately obvious that time per cycle (TC, T_L, T_{NL}) significantly differs between rocking and vibration.

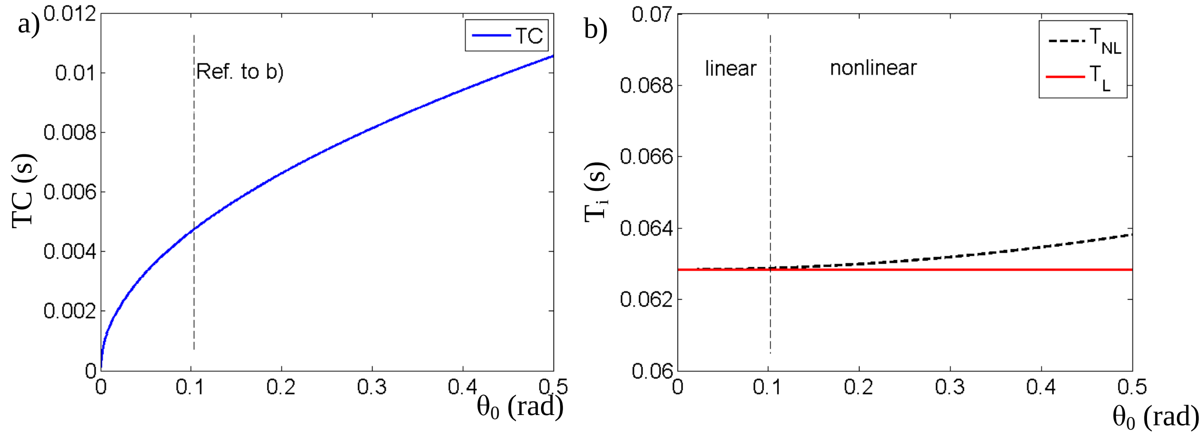


Figure 4: "Period" change versus the initial angle a) for rocking model and b) for linear and non-linear vibration models.

Figure 4 shows the change of "period" with increasing initial angles a) for rocking and b) for a vibrating system. For very small initial angles, as is most likely for a real structure like a building in an earthquake, note that large changes in the time per cycle of the rocking curve (Fig. 4a, Fig. 5a) from one angle to another are observed, while for the vibrating behaviour (Fig. 4b, Fig. 5b) a small or no change in value is shown.

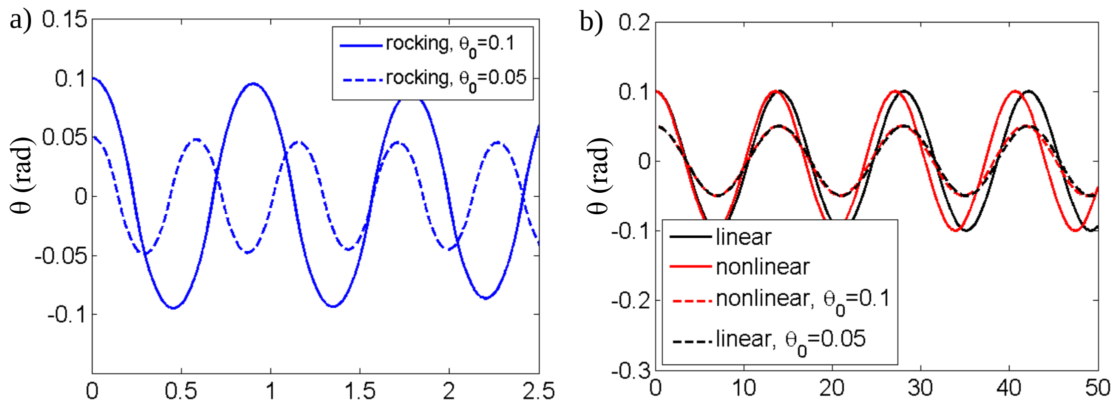


Figure 5: Time series of variable θ for two initial conditions, a) under rocking mode (no energy dissipation), b) under vibration mode (no energy dissipation); compare with Fig. 4

Note further, for a rocking system with θ_0 close to zero, the time for one rocking cycle becomes also zero ($\lim_{\theta_0\to 0}(TC)=0$), as can easily be seen in Fig. 4a as well as Eq. (2-3). For this case the rocking motion stops and the system remains stationary afterward. On the other hand, a vibrating structure continues the motion with a constant period ($\lim_{\theta_0 \to 0}(T_{NL}) = const.$) regardless the value of the initial angle, even for $\theta_0 = 0$, c.f. Fig. 4b and Eq. (2-5).

Another aspect that needs to be considered in the analysis is the change of maximum values per half-cycle. While in a rocking system the reduction of the maximum angle per half-cycle mainly originates from the energy transformation during the impact (physics of impact partners), in a vibrating system it is the damping mechanism that determines the reduction of the amplitudes. Since damping is always present in real structures, the observed decrease in reversal points of an unknown structure could stem from impact when rocking or damping when vibrating. Note that the change in "period" of a damped system subject to free motion is due to two effects. There are (i) the change of peak angle during rocking according to Eq. (2-3) or (ii) the possibility of nonlinear phenomena of the system during vibration according to Eq. (2-6). The rocking or vibration of an unknown structure can be identified by either looking at acceleration recordings (Section 4) or by recording initial angles and change in time per cycle as given in Fig. 4.

4 EXPERIMENTAL OBSERVATIONS OF THE PHYSICAL MODEL

The experiments consist of a series of accelerometer readings and high-speed camera recordings of the physical structure (Fig. 1a). For that purpose a three-axial accelerometer was placed near the lower circumference on the centre line of the end mass of the structure. The experiments are designed to investigate the motion behaviour for different impact partners (ground surfaces) and to evaluate the analytical models against experimental observations. Four different initial lean angles, θ_0, for free dynamic motion (rocking and vibration) are tested on the structure, for six different contact interfaces, which are foam, carpet, black rubber, light felt, steel, and concrete.

Figure 6 shows an acceleration reading of the physical model on the light-felt surface. The uplifts of the structure are immediately observed (as almost constant positive and negative values, respectively) and thus the total dynamical behaviour divides into two regions, (i) where rocking dominates the behaviour and (ii) where the structure vibrates with declining amplitudes.

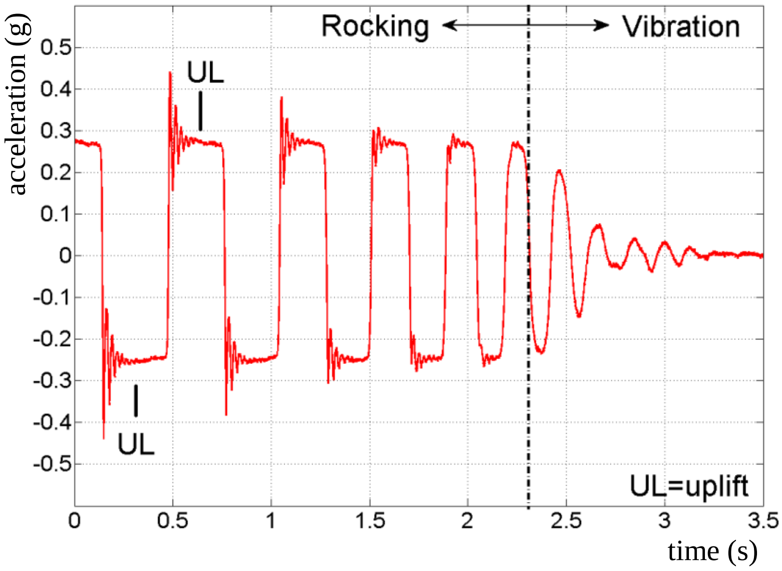


Figure 6: Accelerometer recording of the physical model on a semi-soft (light felt) surface.

Note that for the pure rocking mode, the acceleration characteristic during uplift indicates a free motion by constant values. The value can easily be computed from (2-1) by neglecting θ (since being much smaller than α). Thus, setting θ equal to zero determines the value for the circular acceleration $\ddot{\theta}$ which, multiplied by the distance to the accelerometer, matches satisfactorily with the values observed in Fig. 6. (This work does not aim at finding a quantitative agreement. However, the value of the body-fixed circumferential acceleration direction is computed to be 0.26 g.) The acceleration under pure rocking is thus determined by the system's parameter only and independent of the initial angle and impact partner. Figure 7 confirms this fact as it shows the rocking behaviour of the same system for different impact partners (surfaces of different flexibilities), here on steel (hard) and on carpet (soft).

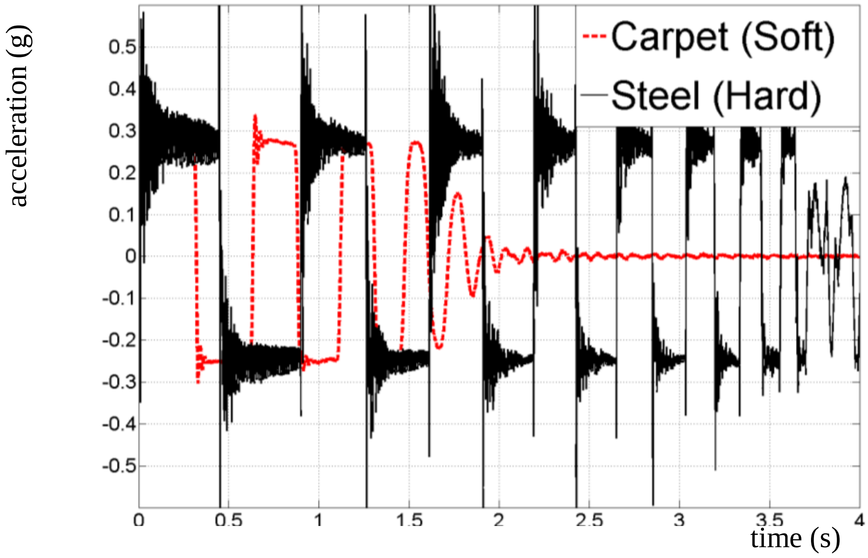


Figure 7: Accelerometer recordings of the physical model on a hard surface (steel) and a soft surface (carpet).

A longer rocking phase is readily observed for the harder surface than for the softer. As previously pointed out, the acceleration values during uplift are the same. A distinct vibrating phase is only recognized for the softer surface. (The supposed vibration phase of the structure on the hard surface is marked by a rather noisy signal, which can be explained by the occurring singularity for θ=0 and a hard impact partner.) This paper does not include high frequency vibrations generated by the impacts.

5 DYNAMIC INFLUENCE OF THE FOUNDATION ON PHYSICAL MODEL

This section discusses multiple experiments to investigate in the influence of the foundation on which the physical model is set. Section 3 revealed that the reduction of reversal points is generated differently under rocking and vibration. It can be readily observed that larger changes in the "period" occur due to the rocking phenomena rather than due to a vibrating mode. In the real scenario the angles are usually very small and thus a structure under vibration can be expected to behave linearly (Fig. 4), i.e. there are almost no changes in the period and the changes of amplitudes are due to the damping mechanism of the soil. On the other hand, a structure that undergoes rocking shows significant changes in "period" and "amplitude" from one quarter-cycle to the next. (It is a quarter-cycle because the rocking motion is free from non-smooth changes during that time. However, in what follows, the times of half-cycles are presented.)

Table 2: **"**Period" (TC) of the physical model on different surfaces in subsequent half-cycles due to free "vibration", $\theta_0 = 5^o$

half-cycle no.	Carpet (C)	Steel (S)	Black Rubber (BR)	Soft Foam (SF)	Light Felt (LF)	Concrete (Co)
1	0.319	0.344	0.328	0.328	0.339	0.350
2	0.315	0.343	0.323	0.330	0.337	0.343
3	0.248	0.290	0.268	0.269	0.285	0.279
4	0.239	0.284	0.259	0.268	0.281	0.268
5	0.187	0.237	0.219	0.219	0.237	0.213
6	0.169	0.236	0.201	0.217	0.228	0.204

7	0.130	0.202	0.169	0.178	0.194	0.163
8	0.113	0.198	0.151	0.174	0.185	0.150
9	0.097	0.167	0.121	0.140	0.156	0.118
10	0.079	0.158	0.122	0.134	0.147	0.108
11	0.068	0.115	0.112	0.105	0.122	-
12	0.049	0.102	0.120	0.097	0.111	-
13	0.057	0.084	0.107	0.083	0.100	-
14	0.013	0.067	-	-	0.086	-
		Key:	Rocking =	Yellow	Vibration =	Orange

Table 3: "Period" (TC) of the physical model on light felt with different initial lean angles.

half-cycle no. / θ_0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
8 degrees	0.478	0.471	0.386	0.380	0.320	0.314	0.269	0.263	0.227	0.217
10 degrees	0.519	0.510	0.405	0.400	0.332	0.327	0.278	0.270	0.230	0.220
12.8 degrees	0.579	0.571	0.439	0.433	0.356	0.349	0.295	0.288	0.245	0.238
half-cycle no. / θ_0	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
8 degrees	0.186	0.176	0.151	0.139	0.118	0.106	0.096	0.081	0.085	0.065
10 degrees	0.186	0.175	0.148	0.136	0.114	0.103	0.093	0.081	0.081	0.068
12.8 degrees	0.203	0.194	0.164	0.153	0.128	0.117	0.101	0.091	0.081	0.077

Table 2 shows the records of the "half-period" changes per half-cycle of the physical model on six different foundations. (The table is colour coded: yellow background denoting rocking and orange marking the vibrating motion.) A difference in the duration of rocking can readily be observed between softer and harder impact partners. While for surfaces such as C, BR, and LF, as defined in Tab. 2 the transition from rocking to vibration occurs around TC $\approx$ 0.15 s, the transition for surfaces S and Co occur around TC $\approx$ 0.1 s. In other words, experiments on surfaces with similar flexibility properties (intuitively observed) predict similar TC values after which transition to vibration takes place. This suggests that rocking is dependent on system parameters and the impact partner, and can be predicted by the value of change in time per half-cycle. Table 3 was obtained in a similar way to that in Tab. 2, for light felt with different lean angles. Here, the TC values after which transitions take place are around 0.15 s regardless the initial angle. This finding would also suggest that the rocking motion of a structure in the real scenario of an earthquake (then under forced motion) is generated for a specific range of occurring frequencies (periods) only – according to our observations for very low frequencies, which is also observed in literature. Furthermore, for larger initial lean angles, the duration of rocking is longer than for smaller ones.
Regardless the observations made above, the underlying mechanism behind the "period" change of a structure undergoing rocking and/or vibration is the minimum rotational energy principle, i.e. how much displacement can occur without liftoff of the structure from the ground.

At last the rate of reduction at each impact is discussed in more detail. The motion of the physical model on six different types of foundations was recorded with a high-speed camera and values of the maximum angles for each half-cycle are evaluated. Figure 8 depicts the angle reduction rates computed as the absolute value of the peak angle in a half-cycle divided by that from the previous half-cycle at impacts for the physical model on C, S, BR, and LF.

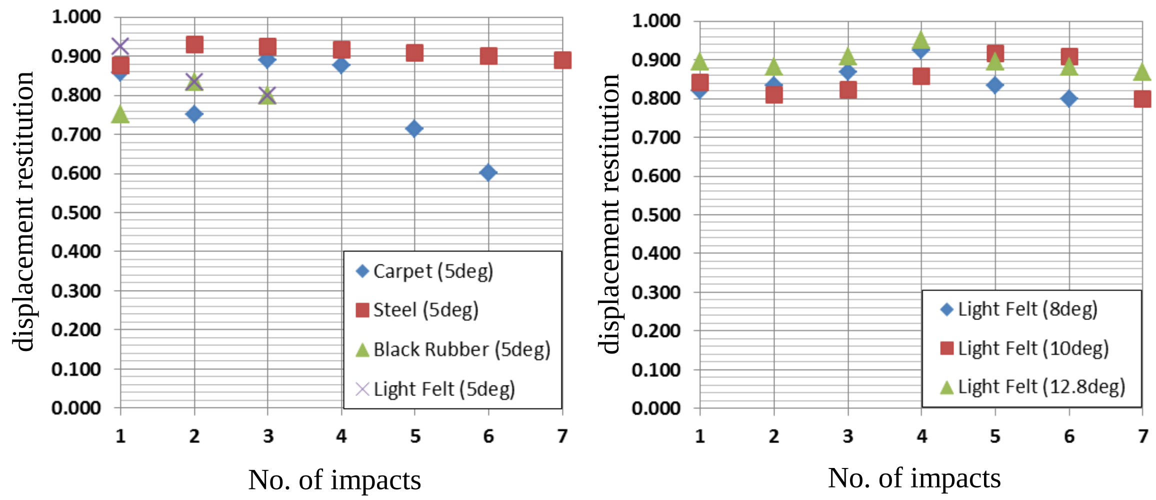


Figure 8: Displacement restitution coefficients of the physical model at each impact a) on four different surfaces at five degrees initial leans, b) on light felt surface at three different initial leans.

It is observed, first, that small deviations occur due to no recognizable pattern and, second, that the overall characteristics follow constant lines. This is not influenced by changing the initial lean angle either, whose effect is shown in Fig. 8. It is noteworthy to communicate that during the experiments small sliding and bouncing effects were present, which could explain the deviations of the restitution coefficients.

According to the gained knowledge of this work an experiment was simulated, describing the rocking and the vibration behaviours separately. Figure 9 shows the comparison of acceleration signals between analytical models and physical structure on light felt.

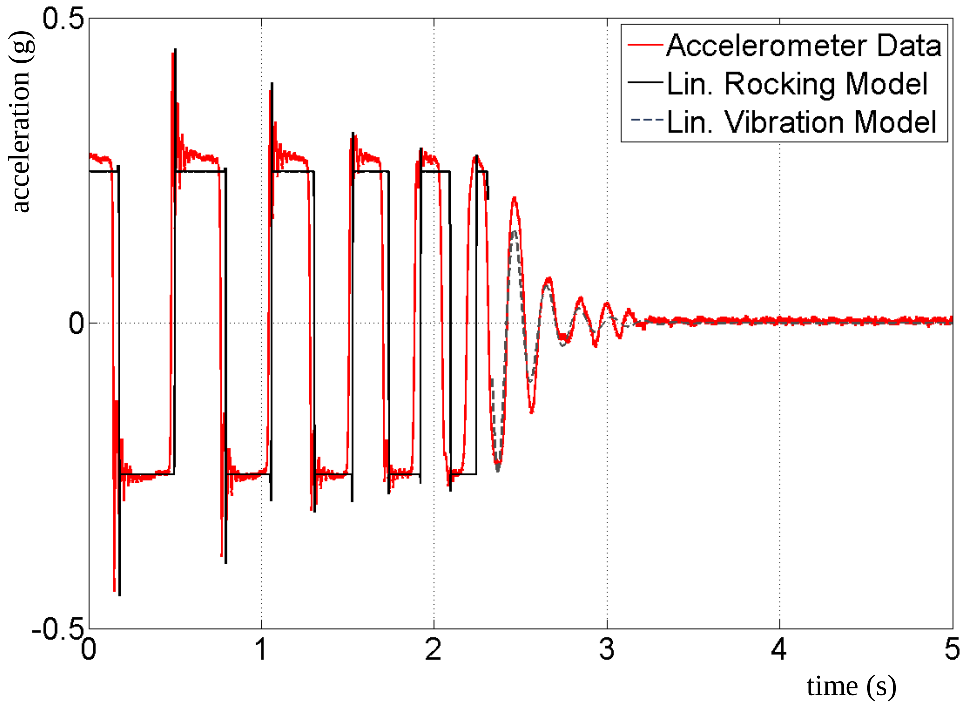


Figure 9: Comparison between accelerometer recording of the physical model on a semi-soft surface (light felt) versus rocking and vibration models.

The simulated rocking behaviour is described by Housner's linearized set of equations (2-1) with θ being neglected. The vibration part is matched separately. It is described by the linear single degree of freedom system (2-2) having added linear damping with a constant damping coefficient to match amplitudes. The analytical rocking model included a constant coefficient of restitution of 0.92. This coefficient matches well with the average value depicted in Fig. 8. The comparison in Fig. 9 shows a satisfying agreement. A more precise quantitative match between theory and experiments can be achieved by studying the impact mechanism in more detail, alongside with carefully designed and systematically performed experimental investigations in future works.

6 CONCLUSIONS

The behaviour of a simple structure deforming in the modes of rocking and vibration was modelled. The modelling methods included (a) a theoretical model with closed form equations describing the response, (b) a theoretical model where the structure response were solved by numerical integration, and (c) an experiment with a simple structure similar to an inverted pendulum on foundations of different stiffness's under free response. It was found that:

1) Under both rocking and vibration, the displacements (angles) decreased. The accelerations for the rocking response are constant and independent of the maximum initial angle in a half-cycle, but under vibration alone they decrease due to energy dissipation. The time for one rocking cycle decreases over time due to a change in the maximum angle in a half-cycle, but under vibration alone, the period was constant for cycles of all amplitudes. The change in rocking response over time is dependent on the impact mechanism and the restitution coefficient. These differences between rocking and vibration mean that both formulations are required if the response of a structure which uplifts is to be reasonably modelled.
2) The experimental study showed that structures under free response are initially dominated by rocking, and later they transition to vibration. For larger initial angles, the duration of rocking is longer than it is for smaller angles.
3) An increase in foundation flexibility decreased the duration of rocking.
4) The rocking and vibrating behaviour of a simple structure under free motion can be qualitatively predicted by simplest, linear rocking and vibration models.

This work is part of a larger study seeking to quantify the response of simple structures under free motion so that their full response can be predicted. A future comprehensive study then includes capturing the dynamic response to forced excitation.

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