# Plastic shear strength of continuous reinforced beams

2008 NZSEE
Conference

Colin Gurley

*TAFENSW Sydney Institute, Australia.* colin.gurley@tafensw.edu.au

**ABSTRACT:** This paper addresses the theoretical, rigid-plastic yield-line collapse-mechanism analysis of continuous, reinforced concrete beams considered as two-dimensional, plane-stress problems including mixed shear/bending mechanisms. It provides a range of collapse mechanisms that may be sufficient for 'exact' analysis using simple calculations of yield segments as free-body static equilibrium problems across the range of ductile-frame beams from long-span, gravity-dominated examples to deep coupling-beams. The analysis assumes that beams are continuous with stronger columns or walls of equal width/thickness thereby avoiding issues related to bearing details. The methods proposed do need to be calibrated to experimental data and, perhaps, they will also assist experimental researchers.

A related issue is that plastic design of steel structures has long been restricted to elements appropriately detailed to ensure ductile failure, for example, by limiting members to sections that are 'compact' enough to prevent premature local buckling of webs or flanges under plastic rotations. Perhaps the plastic design of concrete structures should be more explicitly restricted to elements appropriately detailed for ductility despite the tensile weakness of concrete. This suggests stronger rules for minimum reinforcement content in both directions in the web/mid-depth of concrete beams. This issue is discussed in Appendix A, and Appendix B summarises the relevant suggestions of M.P.Nielsen 1999. Appendix C describes an extension to elasto-plastic analysis where there do seem to be some problems.

## 1 INTRODUCTION

The separate plastic cross-section strengths for beams in bending and in shear were established 1930 – 1970. Alas the (perpendicular) cross-section approach has never been able to provide detailed analysis of the interaction between shear and bending. This has not seemed to matter too much for steel beams perhaps because the effect of bending/shear interaction is usually slight. For concrete beams, the effect is often larger and the ductility of failure, or lack thereof, more significant.

Detailed analysis of interaction would seem to require going beyond the usual skeletal approach (cross-sections are 'small' and plane perpendicular sections remain plane) presumably into the yield-line analysis of beams as plastic two-dimensional plane-stress, 'in-plane' problems. Nielsen 1999 describes the background which largely began with his own work, assisted by others at the Technical University of Denmark, Lyngby from the late 1960s and followed, at ETH Zurich, by Thurlimann, Müller 1976 and Marti 1979.

For earthquake-resistant concrete structures, the usual policy is to ensure that the full cross-section bending strength is realised even at over-strength of main longitudinal reinforcement. This paper supports that policy. Alas earthquake-resistant design inherits the empirical procedures for shear-design that have always been an unsatisfactory feature of general concrete codes. Eurocode 2: 2005 has moved towards plastic design for concrete in shear.

The purpose of this paper is to present a number of prospective rigid-plastic collapse mechanisms that provide for simple calculations, on a pocket calculator or with Microsoft Excel, that are sufficient to ensure that equilibrium is satisfied for each yield segment. Often these mechanisms will suggest 'strut and tie' models but, because they are based on collapse mechanisms, these models will avoid the usually arbitrary character of lower-bound plastic solutions.

A number of mechanisms are required to cover the range from normal ductile frames (20 say > L/d > 4) to deep coupling-beams (4 > L/d > say 0.5). Some of these mechanisms will indeed explain the tension strains in some 'compression' flanges reported by Park and Paulay 1975. Indeed the last example switches the horizontal tension yield in the flange reinforcement to the 'compression' corners (lower-left and upper-right in Fig 21) instead of the tension corners (upper-left and bottom-right).

This paper has a limited objective. It simply aims to exhibit simple (but novel) rigid-plastic collapse mechanisms that provide, at least, a more perceptive understanding of shear-strength. It is intended to complement the excellent experimental work around the world at the Technical University of Copenhagen at Lyngby, at ETH Zurich, the University of Toronto in Canada (Collins 2007) and at the Universities of Canterbury and Auckland. Structural students and structural designers need a strong intuitive understanding of structures and the simple, free-body static equilibrium sketches provided by collapse mechanisms seem a great way to foster that. The mechanisms will, one hopes, lead to a simpler approach to design for shear perhaps after they have been extended to elasto-plastic anaysis.

I am aware of the provisions for diagonal reinforcement in deep coupling beams deriving from research at Canterbury (Park and Paulay 1975). I don't seek to challenge that but rather to provide a basis for calculation which can be extended to include diagonal reinforcement at a later time.

## 2 DOGLEG HINGE: CANTILEVER BEAM WITHOUT COMPRESSION REBARS

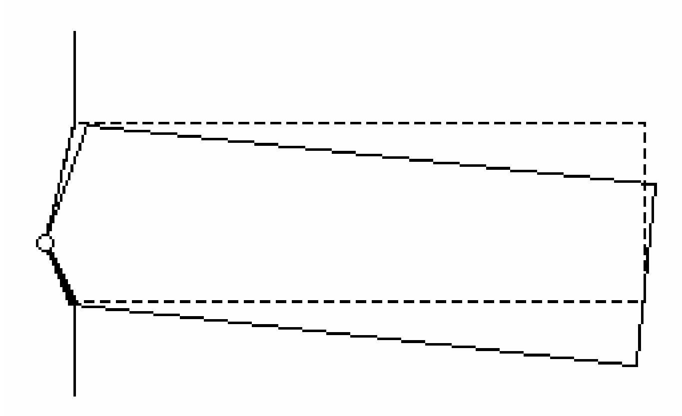

*Fig 1: Dogleg hinge mechanism*

The dogleg hinge of Fig 1 (Gurley 2007) widens established wisdom by assuming that the bending-tension and bending-compression branches do not need to be co-planar. It then becomes possible to shift the rotation centre a short distance inside the face of the support so as to provide a load-path for shear.

Fig 2 shows the forces acting on the yield-segment. The position of the rotation centre is easily calculated so that the diagonal concrete force across the compression yield-line is at the bending compression strength $f_b$ as are the components *C*, *V*. At this stage, it seems appropriate to take $f_b$ at the usual bending compression value $0.85f_c'$. This is probably rather conservative because of the additional restraint to strains (in-plane and out-of-plane) provided by the re-entrant corner.

It seems clear that the dogleg hinge provides an 'exact' solution to this problem. The diagonal strut will intersect the centroid of the load and, theoretically, no transverse reinforcement is required for a load applied along the top of the beam although a minimum content may be required for ductility as discussed in the Appendices.

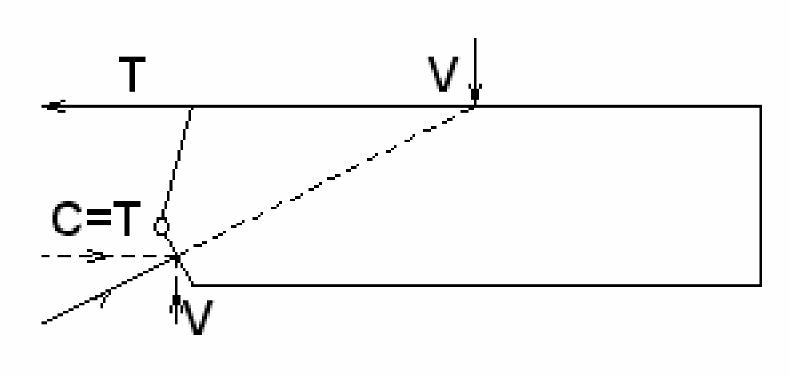


*Fig 2: Static equilibrium of yield-segment*

## 3 DOUBLE DOGLEG MECHANISM WITH COMPRESSION REBARS

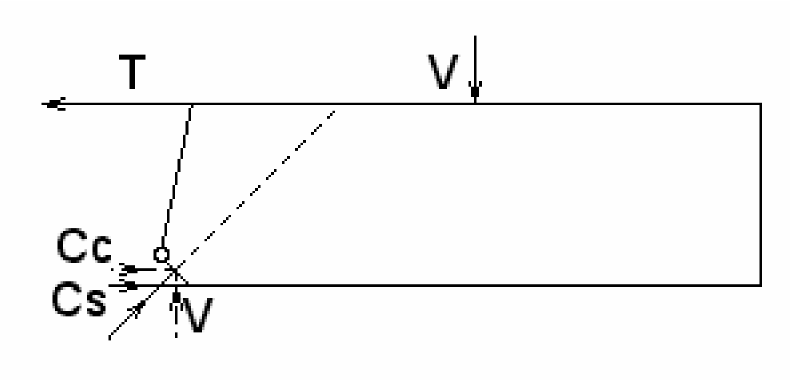


*Fig 3: Compression rebars of strength $C_s$ added*

In Fig 3, the addition of compression rebars (here assumed at zero cover) does not much affect the shear *V* but it does reduce the horizontal component $C_c$ of the diagonal concrete strut. The strut is now steeper and it no longer intersects the centroid of the load. If the full bending strength is to be realised then the compression rebar must be at yield else there would be an incompatible horizontal displacement at that rebar. Fig 4 shows a mechanism that leads to a lesser strength and can be made compatible with non-yielding compression rebars.

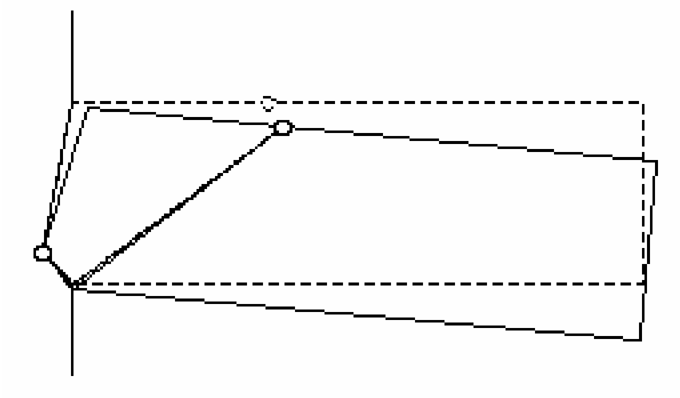

*Fig 4: Double dogleg hinge mechanism*

There is again rotation at the dogleg hinge but also a smaller counter-rotation about a second hinge at the main tension reinforcement. The relative rotations can be calibrated so as to result in zero total horizontal plastic displacement at the compression rebars. The compression rebars need not yield and, to the extent that they do not, the horizontal concrete component $C_c$ will be increased and the collapse shear force *V* and ductility will be reduced.

The yield-line to the second hinge is a bending tension yield-line that cannot be crossed by any diagonal concrete strut. All of the shear must be hung by the transverse reinforcement as in Fig 5 (assuming zero bending tension strength of concrete). The strains across the second yield-line are smaller than those across the first; this does not affect the present rigid-plastic analysis but it may have a larger effect on elasto-plastic analysis. See Appendix C

Fig 5 shows the forces acting on the hanger yield-segment. The rebar compression force $C_s$ is not involved because the hanger segment has only point-contact with the compression rebars. Note that the shear-force transferred is reduced by any loads applied at the top of the beam in the length *y* but not by any loads applied at the bottom. Most loads in laboratory tests are applied at the top but most loads in real buildings are transferred from adjacent slabs and so they are applied near the the bottom of those slabs. It will be safe to treat such loads as applied at the bottom of the beam.

Equilibrium equations for Fig 5 include:

$$C_c\left(D-\frac{k}{2}\right)=V\left(\frac{x+y}{2}\right) \tag{1}$$

$$V-w_t y=rf_{sy}by \tag{2}$$

where: $w_t$ = uniform load at top; $rf_{sy}$ = smeared yield strength of transverse reinforcement; $b$ = width.

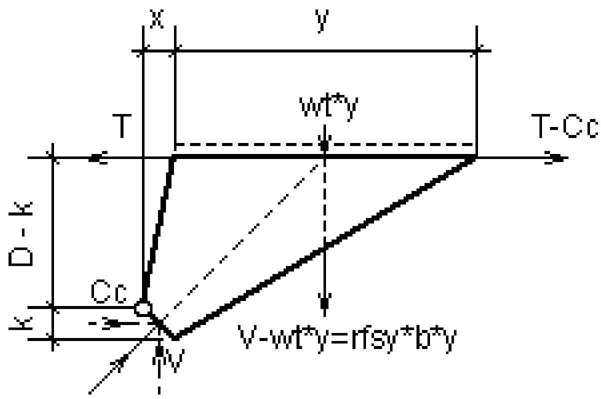


*Fig 5: Static equilibrium of hanger yield-segment*

It is clear that if the smeared strength $rf_{sy}$ of the vertical transverse reinforcement is reduced then:

- The length *y* of the hanger-segment must increase and therefore
- Force $C_c$ must increase and force $C_s$ must reduce so
- The moment strength and ductility across the dogleg hinge will reduce.

If the strength $rf_{sy}$ of the vertical transverse reinforcement is increased then vice versa but, if $rf_{sy}$ is sufficient, then the compression rebars will yield and the full bending-strength will be available.

Strut and tie models can be developed from Fig 5. They will be unique over the yield-length *y* but arbitrary beyond that perhaps including arch segments. If all of the load is applied at the top as $w_t$ then, theoretically there need be no transverse reinforcement beyond the first hanger length *y*. If some load $w_b$ is located at the bottom then there must, at least, be sufficient transverse reinforcement to hang that load. This may be less than the minimum content discussed in the Appendices.

## 4 **LONG-SPAN BEAM UNDER GRAVITY LOAD**

A continuous beam under gravity load differs from the cantilever above only in respect of non-zero moment strength at a midspan hinge. There is no shear at the midspan hinge and so a co-planar hinge is appropriate. Fig 6 shows half of a symmetric beam spanning 8 metres with a full-span uniform load hung from the bottom. Assumed horizontal rebars ($f_{sy}$ = 500 MPa; $f_c'$ = 25 MPa) are:

- Top at ends: 5N32: Ttop = 2011 kN
- Bottom throughout: 3N32: Tbot= 1206 kN; Cbot = 1155 kN
- Mid-depth throughout: 3N12 (including 1 at mid-width) Tweb = 170 kN
- Cc at end = 1025 kN; Yield moment at ends = 1037 kNm
- Cc at midspan = 1376 kN; Yield moment midspan = 603 kNm

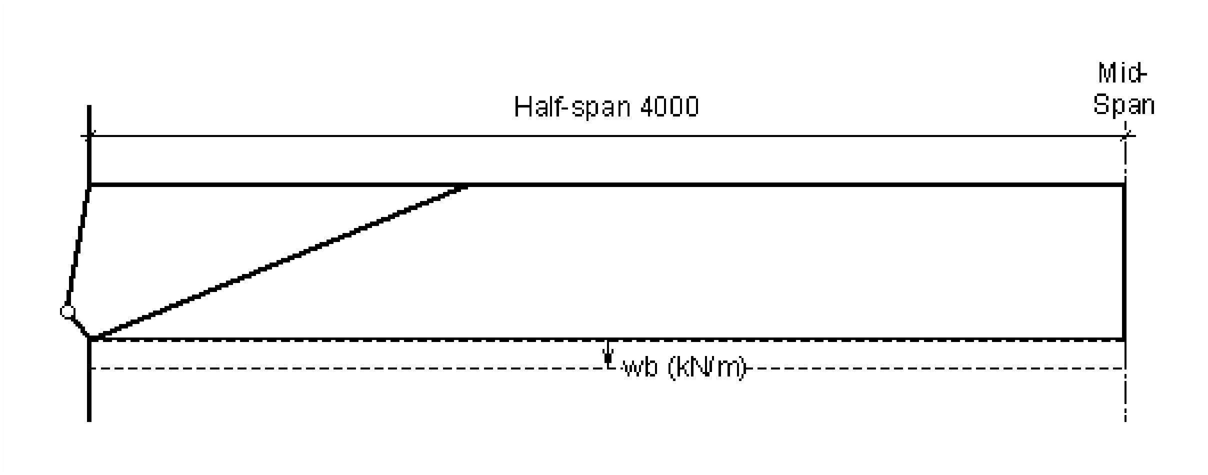


*Fig 6 Half of symmetric 8000 span: 600 deep x 400 wide: 50 cover top & bottom to cg main rebars*

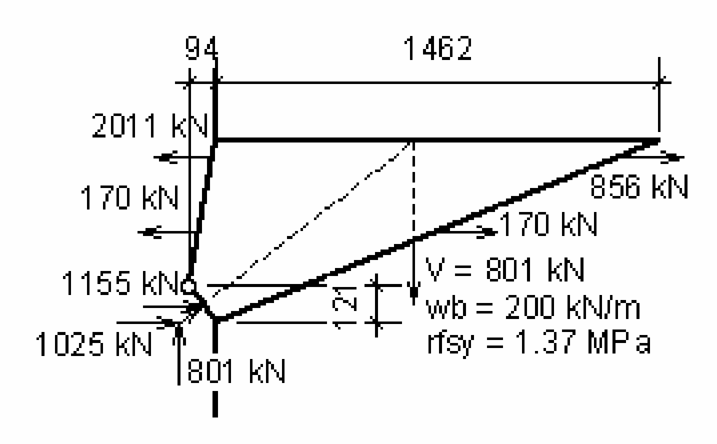


*Fig 7: End segment details*

Fig 7 shows the end segment calculated on the assumption that the smeared yield strength of the shear reinforcement provided in the end segment $rf_{sy}$ = 1.37 MPa is just sufficient to develop the full bending strength. This corresponds to 3N10-215 ties with 3 vertical legs in each tie-set. Heavier end segment ties would not increase overall strength but lighter ties would reduce strength and ductility. If heavier ties are used then the length of the end segment can be reduced to the length required to hang the shear.

Fig 8 shows details for the remaining mid-span segment. The analysis is rigid-plastic so the neutral-axis depth does not include the stress-block depth factor $\gamma$ and is thus about 15% less than the depth calculated on the usual elasto-plastic basis.

The minimum shear reinforcement required to hang the load in the mid-span segment is $rf_{sy} = w_b/b$ = 200 kN/m / 400 mm width = 0.50 MPa which is small but still larger than the AS 3600 minimum content of 0.35 MPa. The dotted lines indicate a lower-bound solution which implies that the tension force in the bottom rebars remains constant over the 2538 length of the half-central segment but this solution is not unique particularly if heavier shear-reinforcement is used.

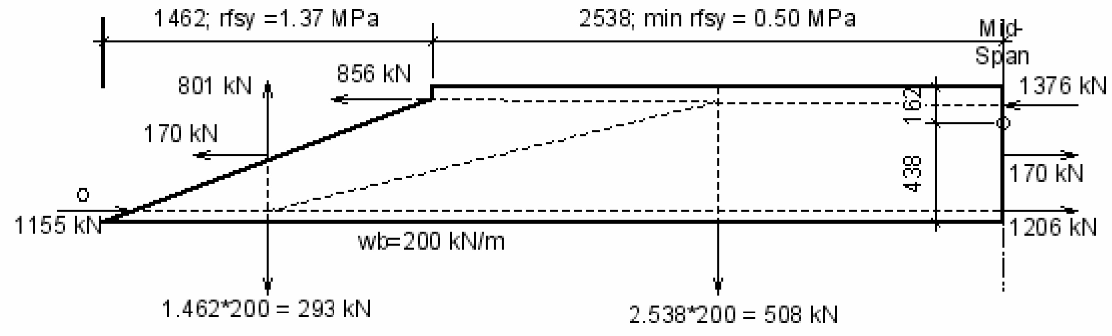


*Fig 8: Midspan details: the dashed lines indicate a lower bound strut-and-tie solution*

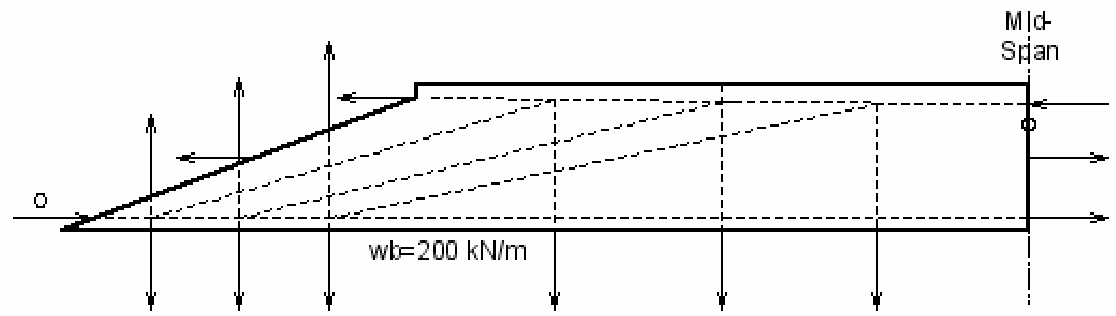


*Fig 9: A non-parallel diagonal compression field*

Fig 9 makes the point that the struts of a diagonal compression-field need not be mutually parallel. In this case, the non-parallelism is caused by the different strengths $rf_{sy}$ of the shear reinforcement in the two adjacent segments but a distributed load along the top of the beam would also contribute to this phenomenon.

A lower-bound solution is re-assuring in that it guarantees the correctness of the corresponding upper-bound solution and can be used to determine rebar cut-off (curtailment) positions. In fact the process of constructing lower-bound solutions can become arbitrary and complicated and, in my view, cut-off positions are better addressed by supplementary upper-bound calculations.

## 5 ANCHORAGE AND DEVELOPMENT LENGTHS

Anchorage and development lengths are defined in the relevant NZS/AS/ACI code of practice. Anchorage is a 3-dimensional issue which has been the subject of much experimental research. The present rigid-plastic theory is a 2-dimensional theory which treats any given layer of rebars as a single entity of the correct total area smeared across the total width of the beam.

It seems obvious that, in engineering practice at least, the codified provisions must govern and that the present theory should not be used to justify any reduction in anchorage lengths. Perhaps in research it may be useful to consider such reductions but there would need to be compelling evidence.

A related issue is that of the tensile strength of the cover-shell concrete outside of the confined reinforcement cage particularly when that shell acts as part (sometimes all) of the bending compression stress-block at plastic hinges. It should be clear that transfer of these bending compression forces back into the confined body of beams necessarily requires transverse tensile stresses in the concrete in the direction of the transverse reinforcement. This is outside the scope of the present theory and all that can be done, within the present theory, to provide a simple and internally consistent approach is to assume that the smeared tensile yield strength $rf_{sy}$ of the transverse reinforcement extends to the outermost compressed edge of the beam although clearly the transverse reinforcement does not.

## 6 CUT-OFF (CURTAILMENT) OF TOP REBARS

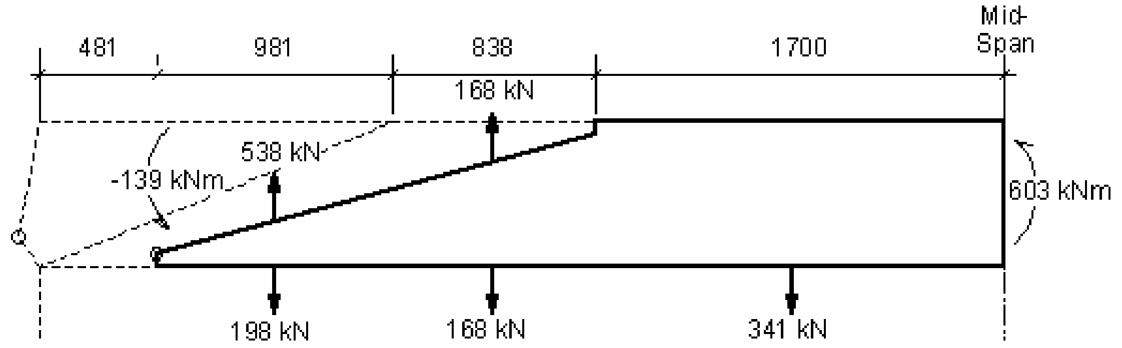


*Fig 10 : Half of 8000 span * 600 deep * 400 wide beam: Midspan dogleg hinge at top rebar cut-off*

Assume that the anchorage for the N32 top rebars is 2300 from the face of support. Fig 10 shows a further dog-leg hinge with the top end positioned at the end of the anchorage. The process of calculating the dogleg hinge is:

- Evaluate the horizontal projection of the tension-branch as the length required to hang the shear-force using just the transverse reinforcement at yield.
- Evaluate the freespan moment of the loads on the segment.
- Calculate the ultimate yield-moment at the dogleg hinge by deducting the moment at the midspan hinge.

Note that the calculation of the moment (of the loads) across the dogleg hinge depends only the horizontal projection and location of the the tension branch but not on the slope or the sense of the moment hogging/sagging. Clearly the rotation centre will shift to a location near the compressed face and the compression branch must be vertical (horizontal stresses).

There is also an issue as to the role of the cover concrete on the tension face. Perhaps the concrete outside the centre of the outermost tension rebar should be disregarded so that the tension-branch finishes on that outermost rebar. The beam 'face' would then step out about 50 mm at the point where the tension-face becomes a compression-face.

Note also that leverarms from the rotation-centre to horizontal rebars are to be measured vertically and so the strains in the rebars are related to distances projected onto a vertical plane. The implication is that the established theory for the ultimate moment (of resistance) of vertical sections remains correct even though the horizontal bending forces are offset horizontally. See Appendix C.

In this particular case, the negative moment of the loads is just 7% of the support yield-moment so the continuing rebars will be determined by minimum content rules.

## 7 8000 BEAM UNDER SEISMIC LOAD

Suppose that the design gravity load drops from a long-term value of 200 kN/m to a short-term value of 150 kN/m coincident with earthquake. Then the length of the part clear-span from (say) left support to midspan positive (co-planar) hinge will increase from 4000 to 4636 resulting in an end-shear of 695 kN and a bearing length decrease from 94 mm to 82 mm. Mechanisms similar to Figs 6 – 10 can be constructed for the yield-segment on the left. The right-hand part clear-span segment will reduce from 4000 long to 3364 with yielding at the midspan coplanar hinge but not at the right-hand support.

## 8 4000 BEAM UNDER SEISMIC LOAD

If the total clear-span is less than 4636 then the beam will not be gravity-load dominated. There will not be a coplanar positive hinge away from supports under seismic load. Instead there will be dogleg hinges of opposite sense at both ends with a downward support shear, say, at the right-hand support. Fig 11 shows the mechanism for a 4000 span otherwise identical to the earlier 8000 span and Fig 12 shows the forces acting on the yield segment.

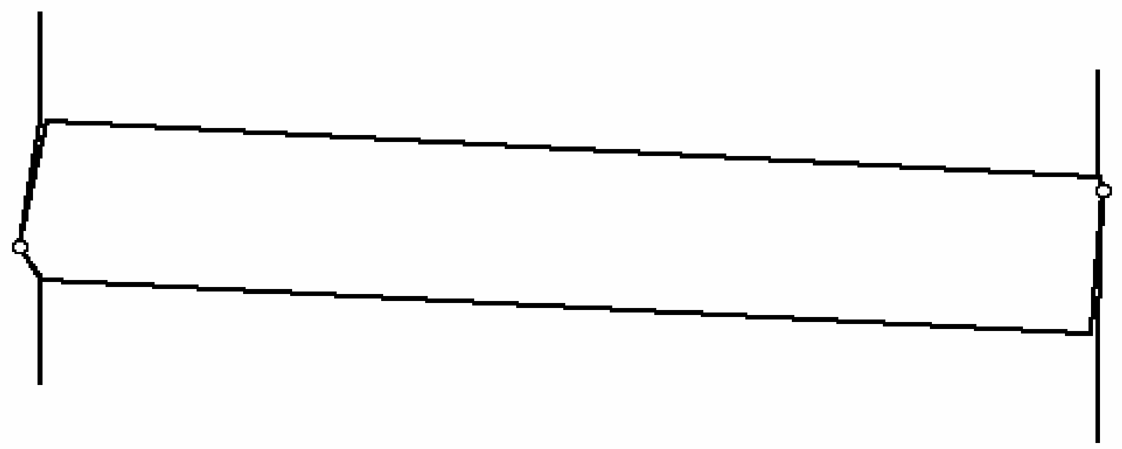

*Fig 11: Mechanism for 4000 beam with single dogleg hinges at each end*

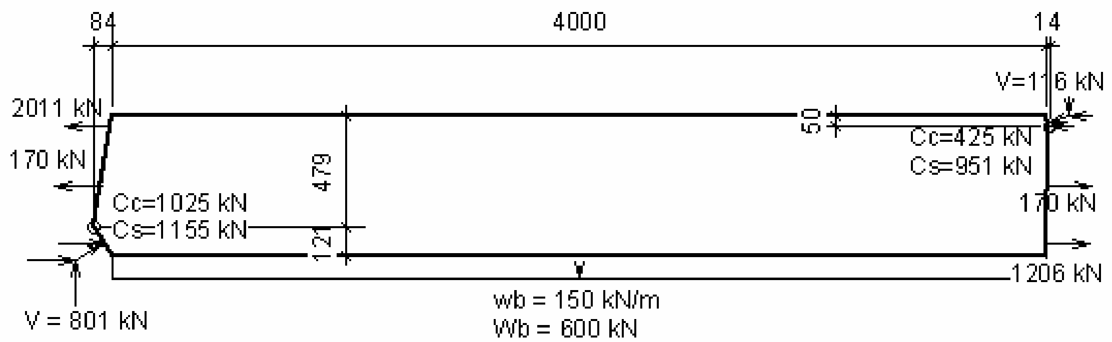


*Fig 12: : 4000 span * 600 deep * 400 wide beam : Free-body static equilibrium for the beam of Fig 11*

The positive (tension bottom) yield moment has increased from 603 kNm in Paragraph 4 (assumes zero midspan compression rebars) to 656 kNm at the right end (assumes 5N32 in compression).

This mechanism and this collapse load will only govern if the transverse reinforcement is sufficient to prevent a lesser collapse load with a double dogleg hinge at the heavier shear.

## 9 4000 BEAM WITH DOUBLE DOGLEG HINGE AT LEFT END

Figs 13 and 14 show an analysis with a double dogleg hinge at the left-hand end. The minimum transverse reinforcement required to sustain the mechanism and load of Figs 11 and 12 has a smeared yield strength of 1.36 MPa over a segment length 1319 from the left end. This transverse reinforcement is about equal to Nielsen's more generous suggestion and corresponds to tie-sets 3N10-217 (3 vertical legs per set at 217 longitudinal spacing). Of course, earthquakes being reversible loads, this reinforcement must be provided over lengths 1319 from each end. The midspan zone needs, at least a smeared yield-strength of 0.38 MPa which is slightly more than the AS 3600 minimum and about half of Nielsen's less generous suggestion. The dashed lines show suggestions for strut and tie solutions; note that the diagonals all slope up to the right because the shear force now has the same sense right across the span.

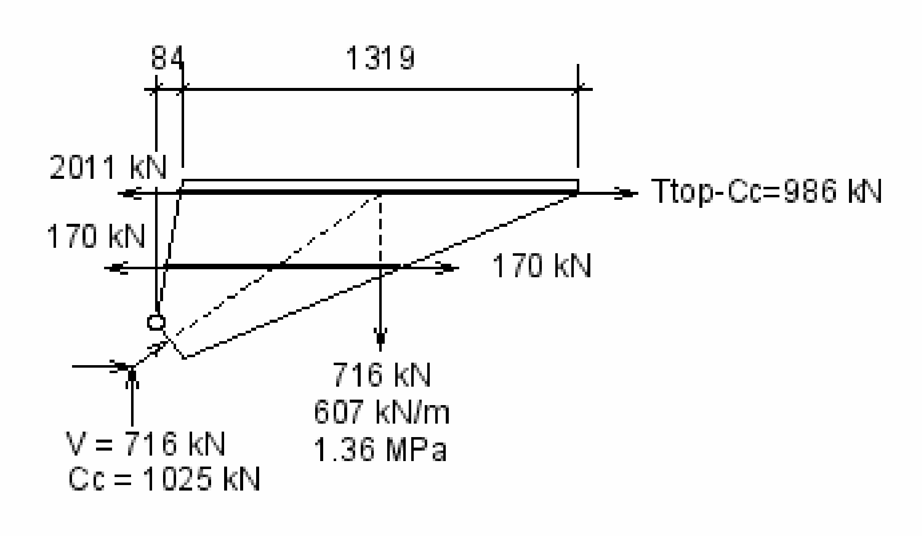


*Fig 13: : 600 deep * 400 wide * 4000 Beam with double dogleg; end hanger segment*

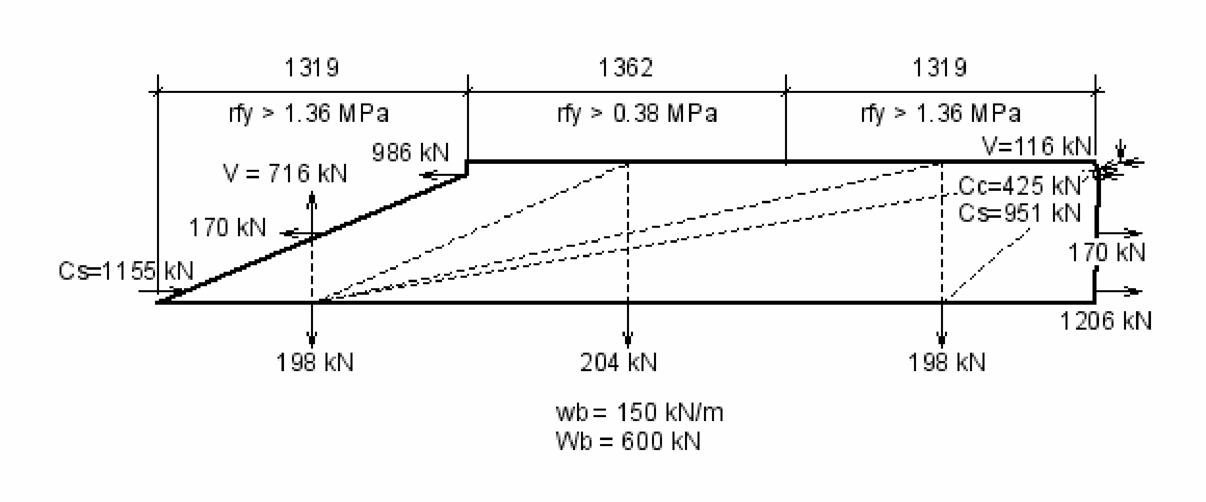


*Fig 14: 600 deep * 400 wide * 4000 Beam with double dogleg; larger segment*

## 10 4000 BEAM WITH REDUCED TRANSVERSE SHEAR REINFORCEMENT

Figs 15 and 16 show analyses which are identical to those of Figs 12 and 13 except that the smeared yield strength $rf_{sy}$ of the transverse shear reinforcement has been reduced 45%:

- From 1.36 MPa (3N10-217) which is the strength required to develop the full bending strength of Fig 12 and, as it happens, about equal to Nielsen's more generous reccomendation
- To 0.75 MPa (3N8-250) which is Nielsen's less generous recommendation but still more than double the minimum content of AS 3600.

The effect of this on static strength is trivial; a 5% reduction in shear at the left-hand support. The effect on ductility parameters is huge:

- A 68% reduction in *Cs* rebar force at left support from yield = 1155 kN to 371 kN
- A 76% increase in *Cc* concrete force at left support from 1025 kN to 1809 kN
- A 76% increase in neutral axis depth at left support from 121 mm to 213 mm

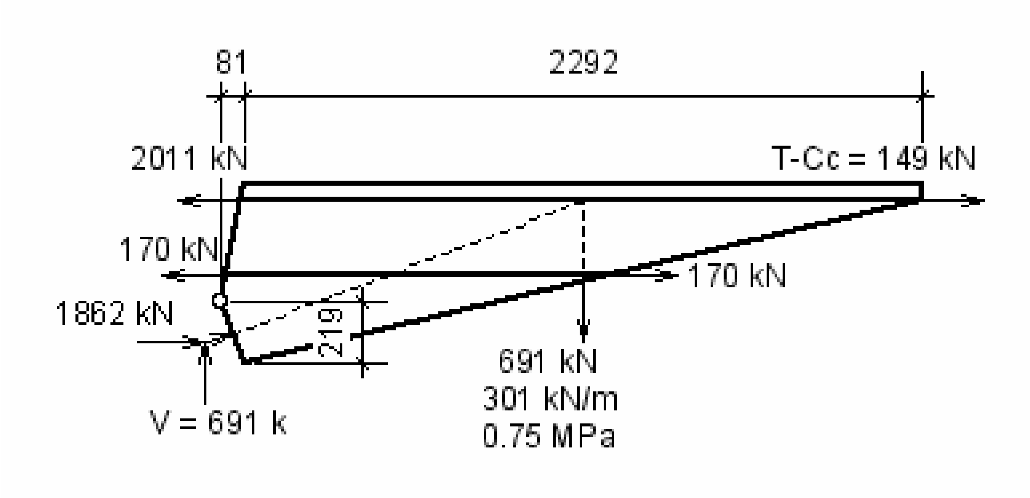


*Fig 15: : 4000 Beam * 600 deep * 400 wide: As Fig 12 but with reduced transverse reinforcement*

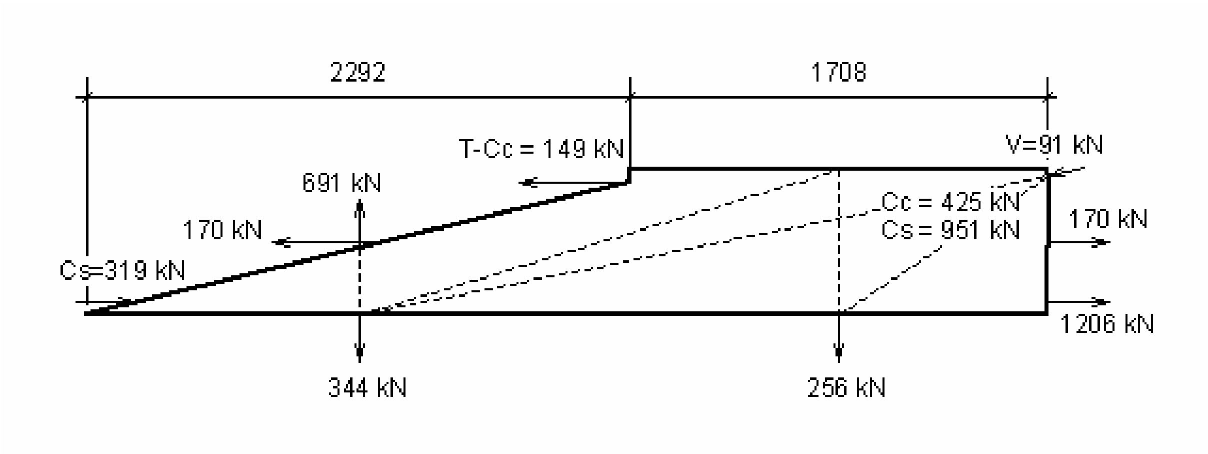


*Fig 16: : 4000 Beam * 600 deep * 400 wide : As Fig 13 but with reduced transverse reinforcement*

The implication is that reducing transverse reinforcement has a much larger effect on ductility than on strength. This does seem to agree, qualitatively at least, with experimental research since the Los Angeles (San Fernando) earthquake of 1971. The point is, of course, that the predictions of this paper are the outcome of simple static equilibrium calculations.

## 11 **1600 SPAN BEAM**

The longer-span mechanisms to date have all been characterised by: $rf_{sy}bL > V$ (3)

meaning that the yield strength of the transverse reinforcement across the whole span is greater than the shear. If this is not so then the yield-line runs along the full-span hypotenuse, all of the shear reinforcement is yielding and, in addition, there is a diagonal concrete strut across the yield-line.

Figs 17 and 18 show a solution for a beam with a 1600 span. There is now a concrete strut crossing the yield-line, although this could have been avoided by increasing the shear reinforcement from 3N10-

250 ($rf_{sy}$ = 1.18 MPa) to 3N12-250. The additional shear is carried across the yield-line by a diagonal concrete strut using the diagonal compressive strength $f_d$ from AS 3600 c12.1.2.2.

We have now reached the sensible limits of a rigid-plastic approach for this particular cross-section, because the neutral axis is close to the mid-depth of the beam-section and the mid-depth rebars would not reach yield on an elasto-plastic analysis. Appendix C describes extension to elasto-plastic analysis.

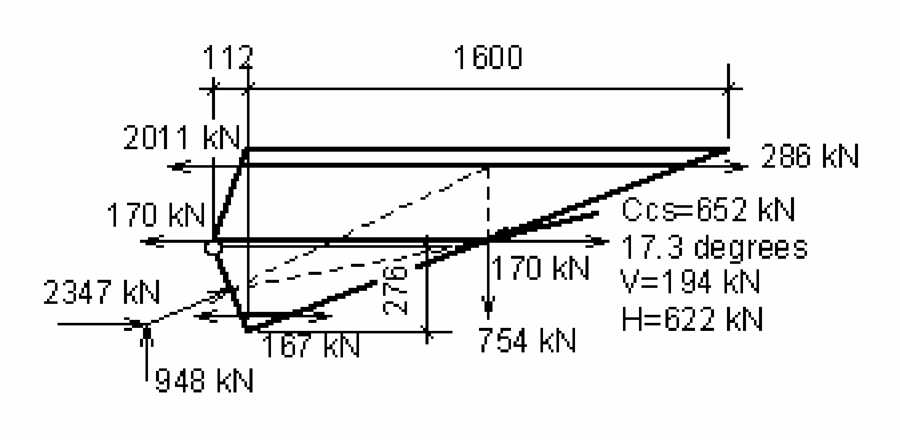


*Fig 17 Upper-left side of : 600 deep * 400 wide * 1600 beam; rfsy = 1.18 MPa = 3N10-250*

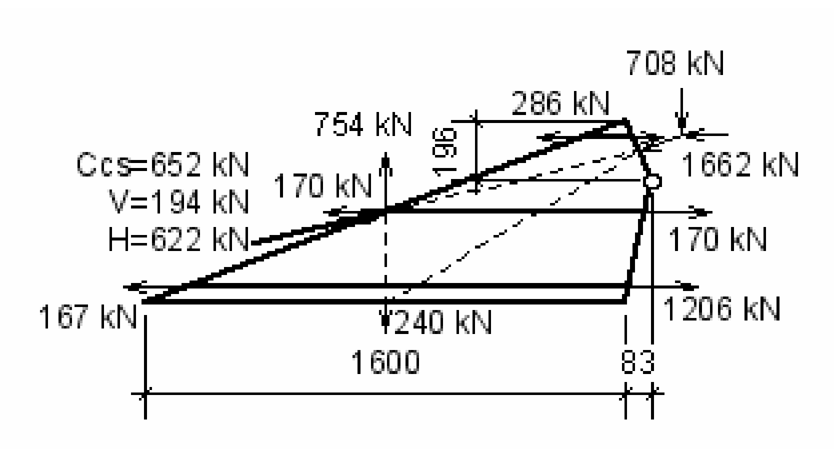


*Fig 18 Lower-right side of 1600 beam*

## 12 HYPOTENUSE MECHANISMS FOR DEEP BEAMS

Fig 19 shows the hypotenuse mechanism introduced in Gurley 2007. There are 2 extra rotation centres indicated by the 2 small circles. Each half yield-segment rotates through the same small angle about the rotation centre located in THE OTHER HALF YIELD-SEGMENT. There is no relative rotation across the hypotenuse yield-line; just displacement. Furthermore, the position of the rotation centres can be adjusted so that there is no relative horizontal displacement at the 'compression' corners (lower-left and upper-right) between, for example, the left support and the lower-right beam segment. There is, again, no need for the 'compression' rebars to yield. The strains on the diagonal yield-line are kinematically related to the strain on the dogleg yield-line.

This is the mechanism for the 1600 span beam described above. The components of the strut force are:

$$C_{csv} = f_d bh\left(\sin^2\alpha - \sin^2\beta\right) \text{ and } C_{csh} = f_d bh\frac{\sin 2\alpha - \sin 2\beta}{2} \qquad (4a,b)$$

where $f_d$ = Diagonal compressive strength; $h$ = hypotenuse length; $2\alpha$ = Angle of hypotenuse from horizontal and $2\beta$ = Angle of zero direct strain across yield-line also from horizontal. Angle of strut from horizontal $\theta = \alpha + \beta$ and attack angle from yield-line $\varepsilon = \alpha - \beta$.

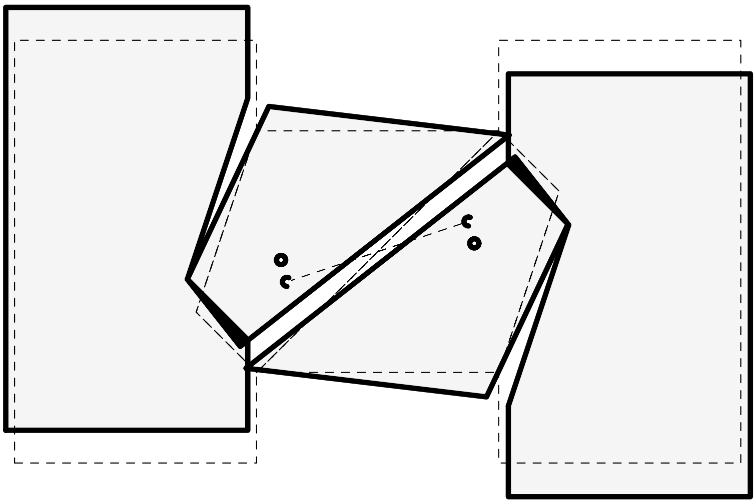

*Fig 19: Hypotenuse mechanism for deep beams*

## 13 **1000 SPAN * 2000 DEEP BEAM**

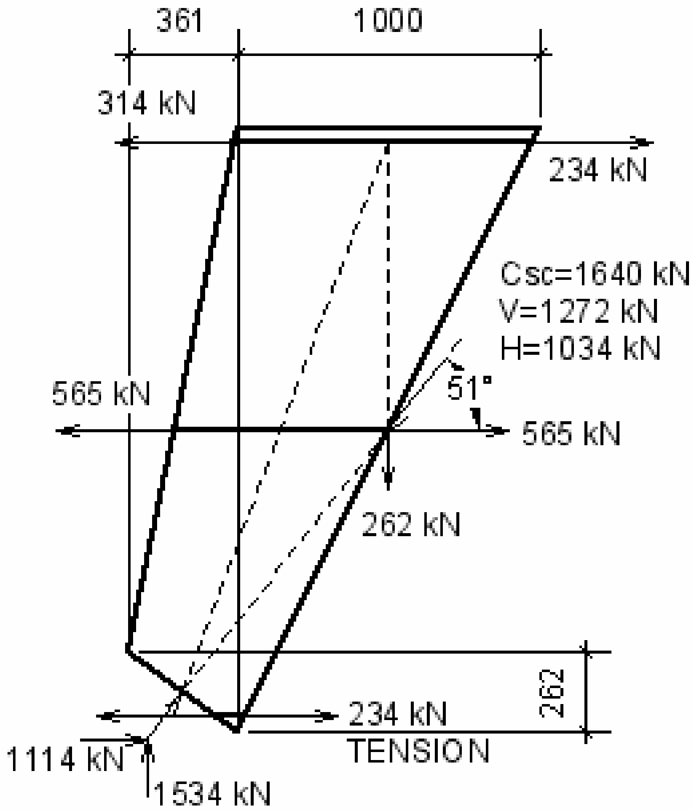


*Fig 20: 2000 * 200 *1000 Span Deep beam: 2N20 T&B; 5N12 Horizontal EF; N10-300 ties; No distributed load*

Fig 20 applies the mechanism of Fig 19 to a deep, lightly reinforced beam without distributed load.

## 14 **SEPARATION MECHANISM FOR DEEP BEAMS**

Finally Fig 21 shows a variation on the mechanism of Fig 20 in which tension yield of the flanges occurs at the 'compression' (upper-right and lower-left) corners. There is no rotation in this case; simply a separation at angle $2\beta$ to vertical. The flange rebars are not (quite) yielding at the 'tension' (upper-left and lower-right) corners and there is some spare moment capacity at the support faces. Equations (4) work here too.

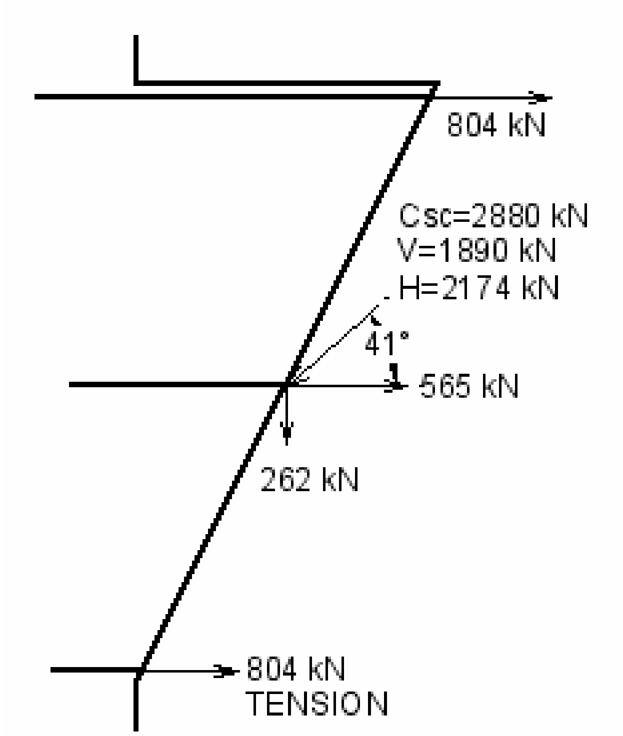


*Fig 21:1000 Span deep beam * 2000 * 200: 2N32 T&B; 5N12 Horizontal EF; N10-300 ties; No distributed load*

## 15 **CONCLUSIONS**

This paper has attempted to continue the theoretical plastic analysis for reinforced concrete beams in mixed bending and shear, building on the work begun by M.P. Nielsen about 35 years ago. Clearly there is a need for experimental verification. I don't have any experimental facilities and never have had, having spent most of my life as a consulting engineer/structural designer. I may be interested to participate 'from afar' in any such work.

The next step is to consider the role of elasto-plastic analysis as against the rigid-plastic analysis of this paper. In the past the difference has been rather trivial but this may not always be true.

I have not yet attempted to consider the effect of diagonal reinforcement. I have no doubt that such reinforcement is often desirable particularly in New Zealand. The mechanisms that I have suggested here will, I think, be equal to the task.

## 16 **REFERENCES**


Collins, Michael P; Bentz, Evan C; Sherwood, Edward G and Xie, Liping 2007. *An adequate theory for the shear strength of reinforced concrete structures*. The Morley Symposium on Concrete Plasticity and Its Application. University of Cambridge, UK, July 2007. See web-site at <http://www-civ.eng.cam.ac.uk/cjb/concplas/>

Gurley, C.R. 1987. *Shear in Reinforced Beams*. Engineers Australia, Civil Engineering Transactions, Vol CE29, No. 2, 1987, pages 96 – 105, Canberra.

Gurley, C.R. 2007. *Core Coupling-Beams in Tall Buildings.* AEES 2007 Conference, Wollongong, Australia, November 2007.

Marti, P, 1979. *Plastic Analysis of Reinforced Concrete Shear Walls*. IABSE Colloqium Kopenhagen *Plasticity in Reinforced Concrete.*

Mueller, P. 1976. *Failure Mechanisms for Reinforced Concrete Beams in Torsion and Bending,* Institute fur Baustatik und Konstrucktion, ETH Zurich, Report No. 65, Reprint from IABSE Memoires, Vol 36-II, Zurich 1976. Birkhauser Verlag Basel und Stuttgart.

Nielsen, M.P. 1999 *Limit Analysis and Concrete Plasticity* CRC Press, Boca Raton.

Park, R and Paulay, T, 1975. *Reinforced Concrete Structures*. John Wiley & Sons

Paulay, T. and Priestley, M.J.N.1992 *Seismic Design of Reinforced Concrete and Masonry Buildings.* John Wiley & Sons.

Standards Australia, *AS 3600 - 2001 Concrete Structures.* Sydney Australia, 150 pp.

## APPENDIX A: MINIMUM REINFORCEMENT AND COMPRESSIVE STRENGTH

Typical relevant non-seismic requirements for minimum reinforcement from AS 3600: 2001 include:

- Sufficient longitudinal tension reinforcement that the cross-section moment strength exceeds 120% of the cracking moment based on bending tension strengths from small specimens; the AS3600 Committee seems not to have considered the possibility that the tensile strength of concrete in real structures might be systematically < the small-specimen value.
- Minimum longitudinal web-reinforcement N12-200 or N16-300 in each face regardless of width but only for beams over 750 deep.
- Minimum transverse shear reinforcement, where required at all, sufficient to equilibrate a smeared tensile yield strength of 0.35 MPa.

Of these three issues, the first seems somewhat generous but the other two both seem too low to ensure ductile failure and hence to warrant plastic design.

The issue of minimum web reinforcement, both ways in side-faces of beams and, perhaps, even distributed across the body of wider beams, is still (or again) an open research issue for general (non-seismic) concrete codes. The best discussion is scattered through 4 chapters (400 pages) of Nielsen 1999. Nielsen's suggestions are clearly tentative but they have had a bearing on Eurocode 2 and they do require significantly more reinforcement than AS3600. Nielsen relates minimum rebar content to the effective compressive strength of concrete using effectiveness factors:

$$f_{cef} = \upsilon f_c' \text{ (2.1.49 on p52)} \qquad \text{(A1)}$$

His effectiveness factors for bending compression strength are, generally, slightly higher than the value 0.85 in the NZS/AS/ACI concrete codes. His effectiveness values for diagonal compressive strength across translation yield-lines range down to 0.25 even with increased mid-depth longitudinal and transverse web reinforcement.

One assumes that Nielsen is not primarily concerned with earthquake resistance although Eurocode 2 will have had to address that risk in Mediterranean countries. It seems that general codes will re-assess these minimum reinforcement provisions but one cannot assume that the outcome of that of will be appropriate for areas of high earthquake risk subject to hysteretic loads.

Codes for high-risk areas such as New Zealand and California will need an ongoing effort to re-assess their position, perhaps initially reviewing experimental data for hysteretic loads in the light of Nielsen's approach. This review should cover effective diagonal compressive strength, minimum reinforcement content and related rules such as maximum spacing. Nielsen's tentative suggestions are summarized in Appendix B.

## APPENDIX B: NIELSEN ON EFFECTIVE STRENGTH AND MINIMUM REBARS

Note that Nielsen uses $f_c$ instead of $f_c$' for cylinder strength.

**Chapter 2: Yield Conditions**

Nielsen relates minimum rebar content to the effective strength of concrete using effectiveness factors: $f_{cef} = \upsilon f_c$ (2.1.49 on p52) (B1)

For low strength concrete in bending (p 368), the effectiveness factors are not much different to the value 0.85 used in the NZS/AS/ACI concrete codes. Nielsen does not seem to mention the gamma-factor $\gamma$ used in some English-speaking (NZS/AS/ACI) codes for reduced stress-block depth.

The more important issue is the diagonal compressive strength across translation yield-lines where the principal tensile strain is (several or many times) greater than the principal compression strain. Figs 17 – 20 relate.

For pure shear and for normal strength concrete, Nielsen proposes:

$$\upsilon_0 = 0.7 - \frac{f_c}{200} = 0.60 \text{ for 20 MPa concrete (2.1.59 page 57)} \qquad \text{(B2)}$$

which compares with the AS 3600 value c12.1.2.2:

$$f_{ccal} = \left(0.80 - \frac{f_c'}{200}\right) f_c' = 0.70 f_c' \text{ for 20 MPa concrete} \qquad \text{(B3)}$$

If there are (or may be ?) ‘initial’ macrocracks at angles > about 45 degrees from the direction of principal compression then diagonal compressive strength is further reduced by a factor for sliding along that pre-existing macrocrack. Nielsen suggests a further reduction:

$$\upsilon = \upsilon_s \upsilon_0 = 0.5\left(0.7 - \frac{f_c}{200}\right) = 0.30 \text{ for 20 MPa concrete (2.1.61 page 60)} \qquad \text{(B4)}$$

**Chapter 4: Disks**

(B2) is discussed on p 264 - 267 and modified to:

$$\upsilon_0 = 0.7 - \frac{f_c}{200} \geq 0.50 \text{ giving a higher result for } f_c > 40 \text{ MPa (4.6.4 page 267)} \qquad \text{(B5)}$$

“Sliding in initial cracks” is further discussed from p 274 essentially confirming (B2). On p 279:

- “To maintain a sliding resistance in cracks, it is important to keep the crack widths small.
- “A necessary condition is that the reinforcement is able to carry a stress in any direction which is larger than the effective tensile strength of concrete.…
- “The effective tensile strength depends on a lot of circumstances….
- “Based on tests and practical experience … the effective tensile strength may be put at half the value measured on standard test specimens”

Nielsen concludes that the minimum smeared yield strength of rebar in each direction should be:

$$rf_{sy} = 0.16\sqrt{f_c} = 0.72 \text{ MPa where } r = \frac{\text{Rebar area}}{\text{Corresponding concrete area}} \text{ see (4.6.17 p 280)} \qquad \text{(B6)}$$

This is more than double the AS 3600 value for minimum transverse reinforcement and corresponds to a rebar content $r$ = 0.144% with 500 MPa rebars.

Nielsen does suggest that (B2) or, presumably (B5), can be used even with sliding in 'initial' macrocracks if the minimum content is roughly doubled to:

$$rf_{sy} = \frac{\upsilon_0 f_c}{8} = 1.5 \text{ MPa for 20 MPa concrete with } r = 0.30\% \text{ (4.6.25 p284)} \qquad \text{(B7)}$$

This is more than four times the AS3600 minimum value for transverse reinforcement!

Nielsen appears to suggest mixed zones within the same design:

- 'Full-strength' zones to (B2 or B5) and (B7) and
- 'Half-strength' zones to (B4) and (B6)

One would need to consider when/whether/how the structure at hand understands this distinction. Perhaps one could reasonably consider lowly-stressed regions as 'half-strength' for seismic design provided that they are definitely protected by a capacity design approach.

Collins 2007 reports that the Canadian code now requires 0.3% of longitudinal mid-depth web reinforcement in deeper beams and that this is sufficient to suppress the 'depth effect' which otherwise reduces the relative shear strength of such beams. Perhaps this is because deeper beams without such reinforcement should be regarded as low ductility (brittle?) elements inappropriate for plastic design.

**Chapter 5 Beams**

At p 400, Nielsen recommends that:

- Small beams (D < 300 to 400) be provided with minimum transverse shear reinforcement to (B6) i.e 0.144% or 0.72 MPa but no longitudinal mid-depth reinforcement.
- Deeper beams up to 1000 should also be provided with longitudinal web reinforcement taken as half of (B6) i.e. 0.072% or 0.36 MPa
- Still deeper beams > 1000 should be provided with tranverse and longitudinal reinforcement both complying with (B6).

Nielsen's Chapter 5 does not seem entirely consistent with his earlier chapters but then, this is a long book written over some years and much of this discussion is clearly tentative. Perhaps it amounts to extending the notion of 'half-strength' zones with effective diagonal strength reduced to (B3): 0.25-0.30 $f_c$. Presumably one could still use a 'full-strength' zone if the design strength required or if, indeed, one considered 'half-strength' zones inappropriate near yield-lines under hysteretic loads.

From p 400, Nielsen discusses the design of beams with 'light' shear reinforcement < (B6) including zero shear reinforcement. This will be relevant to the design of slabs and other elements not part of a primary seismic system but I do not address such issues in this paper.

My own judgement is that Nielsen is correct but too modest in pressing his minimum content rules. I do understand that clients are usually concerned with economical solutions but, ultimately there is, particularly in earthquake-prone regions, a concern for public-safety and for performance even at a small increase in cost. I do conclude that Nielsen's suggestions should be regarded as minimum positions for earthquake-prone regions. Perhaps they do not go far enough?

## APPENDIX C: ELASTO-PLASTIC ANALYSIS

This paper has been limited to rigid-plastic analysis. The extension to elasto-plastic analysis seems the next step.

Fig C1 shows a typical dogleg hinge. The strain at the outermost compression face is:

Strain $e_{cu} = \phi z$ and stress = $0.85 f_c'$ (C1)

where $\phi$ = curvature and the rectangular stress-block is of depth $\gamma z$ where $\gamma$ is the stress-block depth parameter defined in the relevant NZS/AS/ACI code. $\gamma = 0.85$ for $f_c$' <= 28 MPa. The usual limiting value of $e_{cu}$ is 0.003 but perhaps 0.004 after spalling.

Then the strains in the steel rebars are:

Strain $e_{si} = \phi u_i = \frac{e_{cu}}{z} u_i$ and the stresses: (C2)

$$f_s = E_s e_{cu} \frac{u_i}{z} = 600(\text{MPa}) \frac{u_i}{z} \text{ for } e_{cu} = 0.003 \text{ (800 MPa for } e_{cu} = 0.004) \leq f_{sy} = 500 \text{ MPa} \quad \text{(C3)}$$

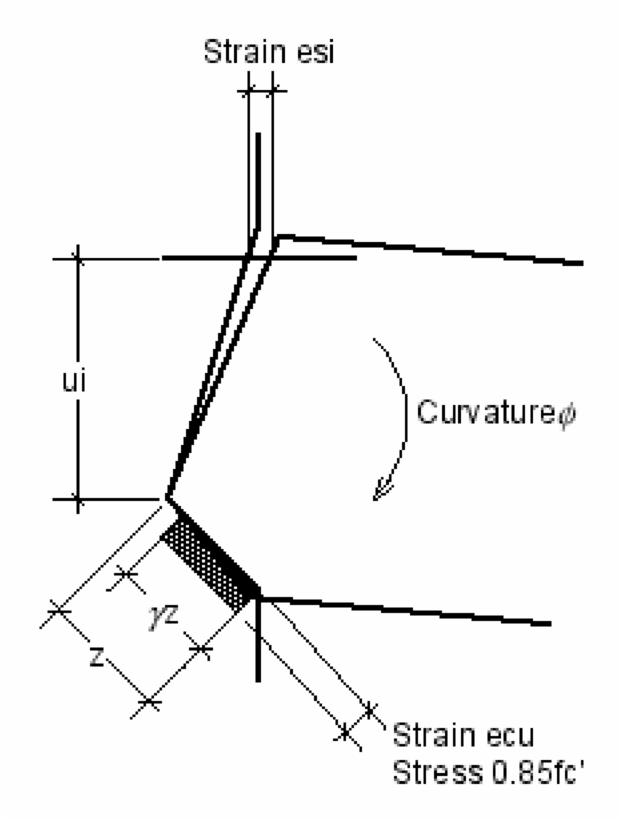


*Fig 22: Strains in a dogleg hinge*

There is also the point that the lever-arms for rebar strains are measured perpendicular to the rebar, that is, vertically for horizontal rebar. Thus the ultimate moment capacity can be calculated without regard to any skew by projecting all of the forces onto a vertical plane.

Unfortunately extending the strain compatibility analysis to the case of the double dogleg hinge may lead to some problems. See Fig 23. The curvature on the left is:

$$\phi_1 = \frac{e_{cu}}{z} \text{ and on the right perhaps } \phi_2 = \phi_1 \frac{k - d_{cs}}{d - d_{cs}} = \frac{k - d_{cs}}{d - d_{cs}} \frac{e_{cu}}{z} \quad \text{(C4)}$$

in order that the compression bars do not yield. But also clearly implying that the strains in the transverse ties are a lot less than the strains in the longitudinal rebars. Are the ties yielding at all? Should this equation be modified so as to include the actual strain in compression rebars?

This is, to the best of my knowledge, the first occasion of yield-lines occurring in pairs with the strains across one yield-line kinematically several times smaller than the strains across the other. The implication may be that there are larger differences between elasto-plastic and rigid-plastic strengths than previously encountered. This discrepancy is not just a problem with upper-bound collapse mechanism analysis. If it indeed exists, then it is an important distinction between rigid-plastic analysis and elasto-plastic analysis and so it also calls into question reliance on lower-bound analysis including the 'strut-and-tie' approach. The upper and lower-bound theorems only apply to rigid-plastic analysis. If the strains in the transverse reinforcement are low then may be a problem for any rational analysis.

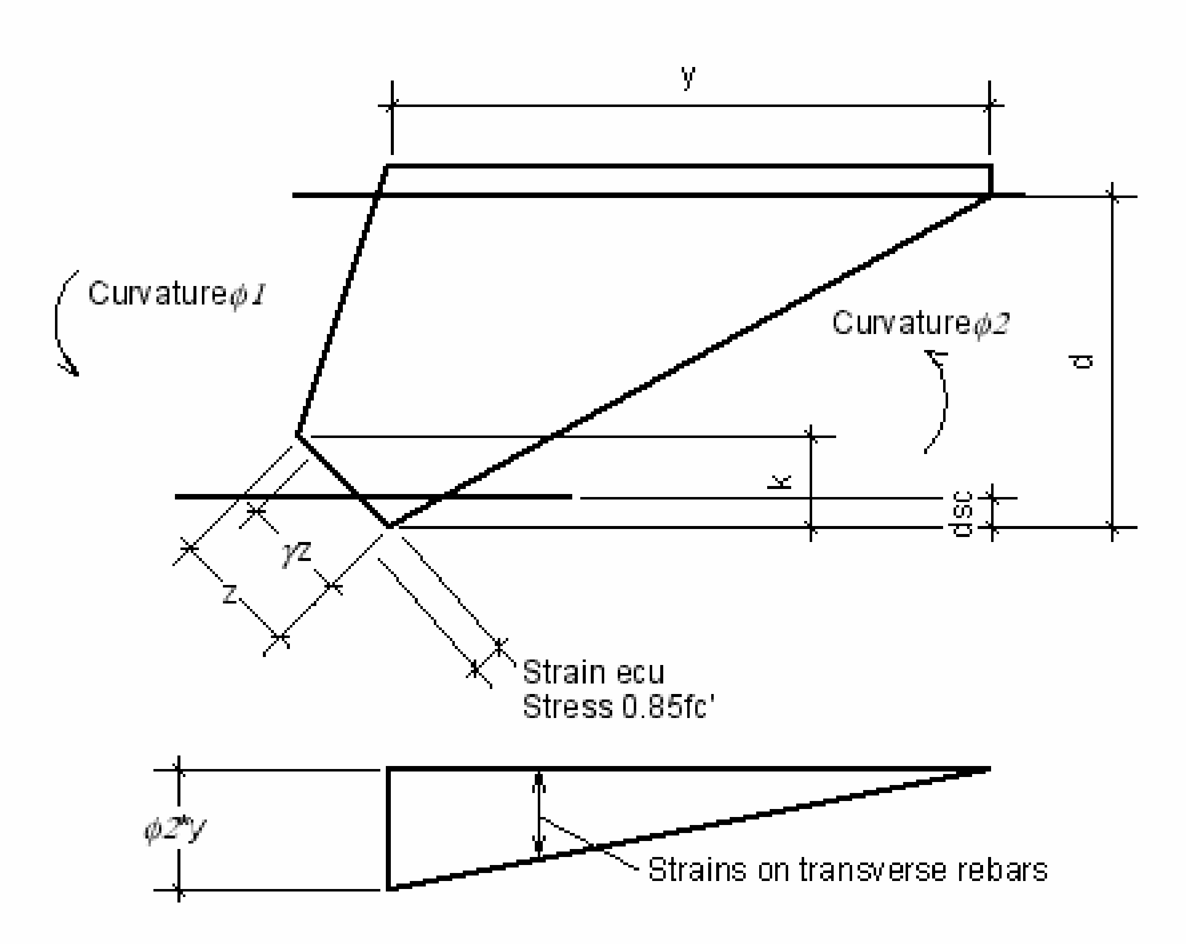


*Fig 23: Strains in transverse ties*

The first point-of-attack here may be the assumption that the maximum concrete strain $e_{cu}$ has the same value as at a midspan hinge, namely, 0.003 at yield or 0.004 after spalling. It seems clear, however that the strain capacity in this re-entrant corner is larger than it would be at a midspan hinge and this value has a proportional effect on the strains in the transverse reinforcement. Clearly there is a need to look at experimental evidence and use the strains in the transverse rebars to evaluate the concrete strain $e_{cu}$.

There is also a need for systematic numeric studies to establish the difference between rigid-plastic and elasto-plastic solutions. If this is consistently less than, say 10% - 20% then practitioners could use the simpler rigid-plastic theory and rely on the protection of load, material and reliability factors.

Figs 17 – 20 all involve cases in which there is a combination of dogleg hinges with translation yield-lines and the rigid-plastic theory does again require that there be definite kinematic relationships between the 2 sets of yield-lines. Once again, it should be possible to evaluate strains on the hypotenuse yield-line relative to the flexural compression value. The example of Fig 21 is different in that there is no rotational yield-line. One can only note that the theoretical principal tension strain is about 6 (actually 5.86) * the principal compressive strain.